24

ACTIVE PORTFOLIO CONSTRUCTION WHEN RISK AND ALPHA FACTORS ARE MISALIGNED

Ralph Karels and Michael Sun
MSCI Consultants

CHAPTER OUTLINE

24.1 Introduction

When looking at active portfolio optimization, there are three main ingredients: The portfolio manager's return forecasts (alphas), a portfolio risk forecast and a risk aversion parameter.

In an ideal world, one factor model should account for both risk and return forecasts. In practice, however, portfolio managers use different models to forecast risk and alpha—in general not necessarily including mutually exclusive but overlapping or related factors such as variations of a momentum factor. These discrepancies may lead to unintended biases in optimized portfolios as the optimizer tends to exploit inconsistencies between the risk and alpha factors resulting in inadvertent bets. Technically, we are talking about risk and alpha factor misalignment when alpha cannot be written as a linear combination of the risk factors. Therefore, there is an alpha part aligned

Rethinking Valuation and Pricing Models. http://dx.doi.org/10.1016/B978-0-12-415875-7.00024-5

with the risk factors and a residual alpha part that is orthogonal to the aligned alpha part. Depending on the portfolio managers' interpretation of the misalignment, it must be treated differently: Residual alpha may contain only noise or may incur risk and return, and hence a positive information ratio. In the first case, residual alpha has to be penalized; in the second case, it has to be managed. It can be shown from an analytical decomposition of alpha into its aligned and residual parts that the optimizer favors residual over aligned alpha in an unconstrained active optimization problem. In principle, there are three ways of mitigation for alpha and risk factor misalignment, distinguished by their focus: (i) Focus on the risk model, (ii) focus on the optimization process or (iii) focus on adding an additional (alpha) factor. We will show that the alpha factor approach creates an unclear risk forecast setting, and furthermore penalizes everything outside the common factor space and thus in particular over-rides the specific risk model; it is not recommended. There is no need to penalize anything other than residual alpha and we will use this approach in the following. We will conclude with three case studies on the penalizing residual alpha approach.

24.2 Framework for Active Portfolio Construction

In general, the goal of active portfolio information is to turn information like expected returns (alphas) into "good" portfolios. The well-accepted portfolio construction theory (Bender, 2008) identifies the relationships between forecast returns, realized returns and portfolio holdings shown in Figure 24.1.

The information coefficient (*IC*) describes the correlation between forecasts and realized returns. The transfer coefficient (*TC*) outlines the correlation between constructed and ideal

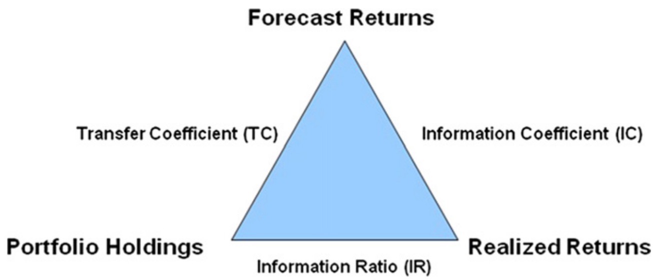

Figure 24.1 Triangle of portfolio construction (Bender, 2008).

(unconstrained) portfolios. The information ratio (*IR*) describes the risk-adjusted returns. In formula terms we thus have:

$$IR_{\mathrm{PF}} = TC \cdot IR_{\mathrm{Opt}}$$
$$IR_{\mathrm{PF}} = TC \cdot IC \cdot \sqrt{Breadth}.$$

The *IR* of the optimal, unconstrained portfolio is the product of the *IC* and the square root of the number of independent bets. It translates into the *IR* of the actual, constrained portfolio by scaling with the transfer coefficient *TC*. Active portfolio construction with return forecasts is characterized by the quadratic optimization problem of the form when the goal is to maximize utility or risk-adjusted return (Grinold and Kahn, 2008):

$$U = \underbrace{\underbrace{r'w}_{\text{Expected Return}} \; \underbrace{- w'(\lambda_{\mathrm{CF}} X^{\mathrm{T}} F X + \lambda_{\mathrm{AS}} D) w}_{\text{Cost of Risk}}}_{\text{Risk Adjusted Return}} \rightarrow \max_{w},$$

where the risk term $X^{\mathrm{T}}FX + D$ stems from a classic multifactor model approach:

$$r_k = x_{k,1} \cdot \hat{f}_1 + x_{k,2} \cdot \hat{f}_2 + \ldots + x_{k,n} \cdot \hat{f}_{\mathrm{n}} + \hat{u}_{k,\mathrm{Spec}},$$

or in matrix form:

$$r = Xf + u,$$

where the excess asset returns r are modeled as a linear combination of the weighted factor returns f plus an idiosyncratic part u. By assuming independence between f and u, and between the individual components of u, we can easily deduce:

$$\mathrm{cov}(r) = X^{\mathrm{T}} F X + D,$$

with X as the matrix of asset exposures to the risk factors f, F as the covariance matrix of risk factor returns and D as the diagonal matrix of volatilities of the asset specific returns u by assumption. Hence, the risk term in this setting is divided into a common factor risk block $X^{\mathrm{T}}FX$ and an asset specific (idiosyncratic) risk block D. By specifying the risk aversion parameters λ_{CF} and λ_{AS} for the respective risk blocks, the portfolio manager can incorporate his/her specific risk and return profile into the optimization process. After specifying an initial portfolio, a benchmark (optional), an asset universe (optional), constraints (optional), expected returns on asset level, and transaction costs (optional) and penalties (optional) the optimization case is complete. If, in the presence of additional constraints, the optimization problem is feasible, the optimizer returns the optimal portfolio with portfolio weights w^* as optimization result.

24.3 Misalignment of Risk and Alpha Models

Active portfolio optimization requires a forecast for both portfolio volatility and exceptional return. In theory, the same factor model would forecast risk and alpha, but in practice different models are used to forecast risk and alpha as portfolio managers intend to incorporate proprietary information not found in the risk model into their alpha forecasts to outperform the market. Mostly the factors used as alpha components or descriptors are not entirely different to the risk model factors, but somewhat overlapping. For instance, different versions or definitions of a momentum, earnings yield, value or growth factor may be used for risk and alpha. Bender *et al.* (2009) show that there is, however, the possibility that these discrepancies between risk and alpha factors may create unintended bets or biases in the optimized portfolios. Before we present a case study of the effects of such a risk and alpha misalignment, let us formalize the meaning of different factor models for risk and alpha by a portfolio manager:

$$r = X_{\mathrm{R}} f_{\mathrm{R}} + u_{\mathrm{R}}$$

$$r = X_{\mathrm{A}} f_{\mathrm{A}} + u_{\mathrm{A}}$$

$$\begin{aligned} \Sigma_{\mathrm{R}} &= X_{\mathrm{R}} F_{\mathrm{R}} X_{\mathrm{R}}^{\mathrm{T}} + D_{\mathrm{R}} \\ \alpha &= X_{\mathrm{A}} w. \end{aligned}$$

Both models above, for excess return r, attribute returns towards specific factors f and an idiosyncratic part u. Moving from returns to risk and incorporating the standard factor model assumptions we obtain:

$$\mathrm{cov}(f, u) = \mathrm{cov}(u, u) = 0.$$

As we have already stated, optimizers tend to exploit inconsistencies between risk and alpha models, resulting in inadvertent and unwanted bets. Lee and Stefek (2008) look at a manager that is betting on a 12-month price momentum factor strategy lagged one month that differs slightly from the momentum factor in the risk model used which has no lag (Figure 24.2).

The alpha model hence includes the return from 13 months ago, but the risk model does not. As a conclusion when optimizing, the optimizer sees return but no factor risk in month 13 and places a large bet. On the other hand, the risk model includes

Figure 24.2 Different momentum factors in the risk and alpha model (Lee and Stefek, 2008).

return from one month ago, but the alpha model does not. Similarly, the optimizer sees factor risk but no return for one-month momentum and places a negative bet here. The combination of these two effects is probably not what the manager had in mind when designing his alpha model. When the risk and alpha models match their definition of momentum, the exploitation of these inconsistencies by the optimizer resulting in concentrated bets disappears.

Let us now have a closer look at the meaning of risk and alpha factor misalignment. Per definition, risk and alpha factors are aligned if alpha is spanned entirely by the risk factors, i.e. alpha can be written as a linear combination of the risk factor exposures:

$$\alpha = \sum_j \gamma_j X_{R_j}.$$

If alpha on the other hand cannot be written as a linear combination of the risk factors, then there is a residual part of alpha that is orthogonal to the alpha part that is aligned with the risk factors. In a stylized graphical representation, the alpha decomposition looks like Figure 24.3. The question arising now is: Is alpha and risk misalignment to be avoided generally? Since the alpha part not aligned with the risk model may contain not only noise, but also may incur return and risk and hence a positive information ratio IR, the answer to this question is no. There are

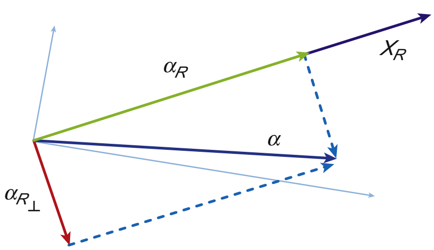

Figure 24.3 Decomposition of alpha into a part aligned with the risk model and a part orthogonal to it.

in fact cases where the return–risk tradeoff of the non-aligned alpha should be managed.

Let us now look closer into the analytical decomposition of alpha along the lines of the sketch above. When regressing alpha against the risk factor exposures, the fit from the regression is represented by the spanned alpha α_{R} and the residual is represented by the orthogonal alpha $\alpha_{\mathrm{R}_\perp}$. Therefore the managers' alpha may be decomposed into a part that is spanned by the risk factor exposures and a part that is orthogonal to them (Bender *et al.*, 2009):

$$\alpha = \underbrace{X_{\mathrm{R}}\left(X_{\mathrm{R}}^{\mathrm{T}}X_{\mathrm{R}}\right)^{-1}X_{\mathrm{R}}^{\mathrm{T}}\alpha}_{=\alpha_{\mathrm{R}}} + \underbrace{\left(I - X_{\mathrm{R}}\left(X_{\mathrm{R}}^{\mathrm{T}}X_{\mathrm{R}}\right)^{-1}X_{\mathrm{R}}^{\mathrm{T}}\right)\alpha}_{=\alpha_{\mathrm{R}_\perp}}.$$

Note that the components of alpha are viewed differently by the risk model:

$$X_{\mathrm{R}}^{\mathrm{T}}\alpha_{\mathrm{R}} = X_{\mathrm{R}}^{\mathrm{T}}X_{\mathrm{R}}\left(X_{\mathrm{R}}^{\mathrm{T}}X_{\mathrm{R}}\right)^{-1}X_{\mathrm{R}}^{\mathrm{T}}\alpha = X_{\mathrm{R}}^{\mathrm{T}}\alpha$$

$$X_{\mathrm{R}}^{\mathrm{T}}\alpha_{\mathrm{R}_\perp} = X_{\mathrm{R}}^{\mathrm{T}}\left(I - X_{\mathrm{R}}\left(X_{\mathrm{R}}^{\mathrm{T}}X_{\mathrm{R}}\right)^{-1}X_{\mathrm{R}}^{\mathrm{T}}\right)\alpha = 0.$$

A tilt in the direction of α_{R} incurs a factor risk but a tilt in the direction of $\alpha_{\mathrm{R}_\perp}$ incurs no factor risk since the orthogonal alpha is outside the risk factor space.

24.4 Portfolio Optimization with Alpha Decomposition

Let us now examine the effects of the alpha decomposition above on an unconstrained active portfolio optimization problem:

$$\alpha^{\mathrm{T}}w - \frac{\lambda}{2}w^{\mathrm{T}}\left(X^{\mathrm{T}}FX + D\right)w \rightarrow \max.$$

Since there are no constraints involved here, the optimal portfolio is easy to obtain:

$$w^* = \frac{1}{\lambda}\left(X^{\mathrm{T}}FX + D\right)^{-1}\alpha.$$

If we further assume for simplicity reasons that all assets have the same specific risk σ_s and use the alpha decomposition derived in the previous section, we obtain (Lee and Stefek, 2008):

$$\Sigma = X_{\mathrm{R}}F_{\mathrm{R}}X_{\mathrm{R}}^{\mathrm{T}} + \sigma_s I$$

$$w^* = \frac{1}{\lambda\sigma_s^2}\left[\alpha_{R_\perp} + \left(1 - X_R\left(X_R^T X_R + \sigma_s^2 F_R^{-1}\right)^{-1} X_R^T\right)\alpha_R\right].$$

Close inspection of the optimal portfolio weights above reveals that the optimizer favors $\alpha_{R_\perp}$ over α_R. While the orthogonal alpha is only scaled to adjust for specific risk, the spanned alpha is also scaled to adjust for specific risk but additionally twisted and shrunk to adjust for common factor risk. In the special case of a risk model with just one single factor f_R with volatility σ_{f_R} this phenomenon becomes even more obvious. Since the matrices $F_R = (\sigma_{f_R})$ and $X_R = (x_R)$ now are 1×1, the necessary inversions are easy to calculate and we obtain:

$$w^* = \frac{1}{\lambda\sigma_s^2}\left[\alpha_{R_\perp} + \left(\frac{\sigma_s^2}{n\sigma_{f_R}^2 + \sigma_s^2}\right)\alpha_R\right].$$

The equation suggests the following. (i) Since the ratio in brackets is always smaller than 1, $\alpha_{R_\perp}$ always has a greater weight than α_R. (ii) As the number of assets n and the factor volatility σ_{f_R} increases, the weight in the part of alpha spanned by the risk factor exposures decreases even more, thus increasing the size of the unwanted bets in the optimal portfolio. There are several ways to mitigate this effect, focusing on the risk model or the optimization itself. We will explore those in Section 24.5.

24.5 Mitigation for Alpha and Risk Factor Misalignment

One way to mitigate the effects of alpha and risk factor misalignment is to focus on the risk model. We will assume for now that risk and alpha factors are not mutually exclusive, but more a like in practice, related to each other as shown in the momentum example in Section 24.3. Those risk model factors resembling alpha model factors are called related risk factors. The following approaches (Lee and Stefek, 2008) may be applied in order to reduce misalignment between alpha factors and their risk model counterparts with increasing steps of complexity.

(i) Drop the related risk factors from the model by setting all asset exposures to those factors to zero.

(ii) Modify the risk model by swapping all related risk factors by their alpha factor counterparts and retain all other risk factors. That implies building and estimating a new risk model including the new alpha factors.

(iii) Alter the risk model by swapping the related risk factor exposures X_{R_i} with the alpha factor exposures X_{A_i}.

(iv) The portfolio manager may use his/her risk model to emulate the new risk model obtained in (ii). The idea here is to approximate the covariance matrix of a risk model based on the portfolio managers' alpha factors and the retained risk factors. The emulated model would be of the form $\Sigma = X_C F_C X_C^T + \Delta_R$, where F_C represents the approximation of the covariance matrix of the new model's factor returns and X_C are the exposures to the retained risk and alpha factors. The diagonal matrix of specific return volatilities Δ_R stems from the managers' risk model.

Another way to mitigate issues arising from misaligned risk and alpha factors is to focus on the optimization process itself. The idea is to penalize the portion of the alpha not related to the risk factors (the orthogonal alpha α_R) to counteract the tendency of the optimizer to over allocate this part (Bender *et al.*, 2009). This technique is called "penalizing residual alpha" and implies adding a quadratic penalty term for residual alpha and an adjustment parameter ϑ to the utility function in the optimizer:

$$\alpha^T w - \frac{\lambda}{2} w^T \Sigma w - \vartheta \left(w^T \alpha_{R_\perp} \right)^2 \rightarrow \max.$$

The parameter ϑ allows the portfolio manager to control the importance of the penalty term similarly to λ adjusting for the importance of the risk term. How should this parameter be chosen? We consider two cases:

(i) The residual alpha is essentially noise. This is the case when the managers' alpha and the risk model factors capture essentially the same properties, but measure them slightly differently (the momentum case). Then the manager should choose ϑ sufficiently large to avoid any tilt towards $\alpha_{R_\perp}$.

(ii) There is some factor return and risk associated with a tilt towards the residual alpha. In this case, the manager should choose $\vartheta \sim \lambda \sigma^2_{\alpha_{R_\perp}}$ to achieve an optimal risk–reward tradeoff. Choosing ϑ like this associates the proper risk penalty with a bet on the orthogonal alpha $\alpha_{R_\perp}$. We assume here that residual alpha return is uncorrelated with the risk model factor returns.

A third way of possible mitigation of alpha and risk factor misalignment is to add an additional factor to the risk model and penalize it (Saxena and Stubbs, 2011):

$$\alpha^T w - \frac{\lambda}{2} w^T \Sigma w - \vartheta w^T \left(I - P_{X_R} \right) w \rightarrow \max.$$

With $P_{X_R} := X_R(X_R^T X_R)^{-1} X_R^T$ and $w_\perp := (I - P_{X_R})w$ we easily observe symmetry and idempotency:

$$P_{X_R}^T = P_{X_R}$$
$$(I - P_{X_R})^2 = I - P_{X_R},$$

and therefore can rewrite the utility optimization problem:

$$\alpha^T w - \frac{\lambda}{2} w^T \Sigma w - \vartheta w_\perp^T w_\perp \to \max.$$

The so-called alpha factor $I - P_{X_R}$ represents the orthogonal part of implied alpha. Now there are two risk terms in the utility function that we cannot combine—hence the risk setting in this approach is unclear. Also, the meaning of the penalty parameter ϑ and how to calibrate it are not straightforward. Furthermore, this approach penalizes everything outside the common factor space and would thus especially over-ride the specific risk part that cannot be desired by the portfolio manager. There is no need to penalize anything other than residual alpha. Thus, we do not recommend this mitigation of alpha and risk factor approach. If there is doubt about the risk model missing factors with high explanatory power, the factors should be added directly to the model. If the missing factor is the orthogonal alpha $\alpha_{R_\perp}$, the penalizing residual alpha approach adds the missing factor directly to the risk model as desired. When we introduce $\alpha_{R_\perp}$ as a new column to the exposure matrix X_R and rename it Y and also adjust the factor return covariance matrix F_R according to (Stefek, 2007):

$$Y = \begin{bmatrix} X_R & \alpha_{R_\perp} \end{bmatrix} \qquad F = \begin{bmatrix} F_R & 0 \\ 0 & \frac{\theta}{\lambda} \end{bmatrix},$$

we obtain as the new enhanced utility maximization problem:

$$\alpha^T w - \frac{\lambda}{2} w^T (YFY^T + \Delta) w \to \max.$$

Simple matrix algebra now shows that this translates into:

$$\alpha^T w - \frac{\lambda}{2} w^T (X_R F_R X_R^T + \Delta) w - \vartheta \left(w^T \alpha_{R_\perp}\right)^2 \to \max,$$

which represents exactly the penalizing residual alpha approach. This shows not only that contrary to the alpha factor approach discussed above, the risk forecast $YFY^T + \Delta$ in this setting is clearly defined, but also that if the missing factor in the risk model is the orthogonal alpha, penalizing residual alpha is a superior approach to the alpha factor approach.

24.6 Case Studies

In a recent paper, Bender *et al.* (2009) illustrated how penalizing the residual alpha in portfolio optimization may improve a portfolio's exposures and *ex ante* information ratio. We will briefly discuss this result as a modifying objective function by including a penalty term on the residual alpha that helps to mitigate the mismatch between alpha and the risk by assigning a suitable penalty to the residual alpha.

In Figure 24.4 we plot four portfolios' exposures to the spanned and residual alphas. It is immediately clear that the parameter θ affects the portfolio's tilt on the spanned ($\boldsymbol{h}^{\mathrm{T}}\alpha_{\mathrm{R}}$) and residual alpha ($\boldsymbol{h}^{T}\alpha_{\mathrm{R}_{\perp}}$). When the penalty term is neglected the resulting portfolio has a significant tilt on residual alpha. Gradually increasing the value of θ will tilt the optimal portfolio's exposure away from residual alpha until there is no exposure left, e.g. in the case of when $\theta = 0.01$.

The authors further analyzed the cases where the residual alpha is a noise, and contains return and risk (Figure 24.5). In the former case where alpha (momentum alpha, which is defined as $\alpha_t = r_{t-2} + r_{t-3} + \ldots + r_{t-13}$) is the "true" alpha and by assumption the risk model is estimated without error (within which the momentum risk factor is defined as $X_t = r_{t-1} + r_{t-2} + \ldots + r_{t-12}$), tilt on the momentum factor only contributes risk but generates no return. Figure 24.5 demonstrates that increasing θ for penalizing the residual alpha helps the optimized portfolio to achieve a higher information ratio.

In Figure 24.6 we again look at the same alpha and risk model, but assume the residual alpha contains return and risk. In this

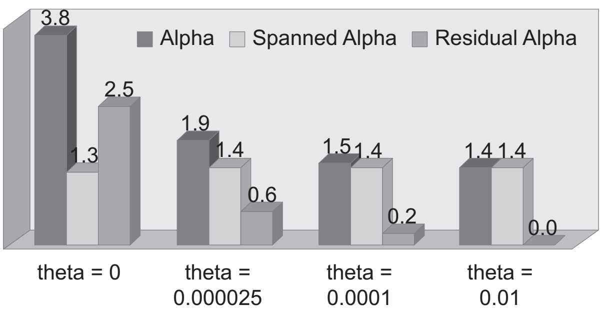

Figure 24.4 Exposure to alpha components.

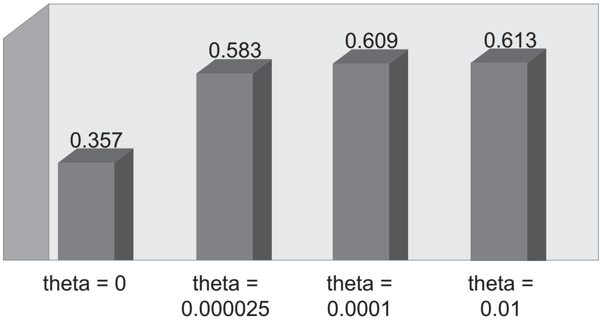

Figure 24.5 Information ratio when residual alpha is noise.

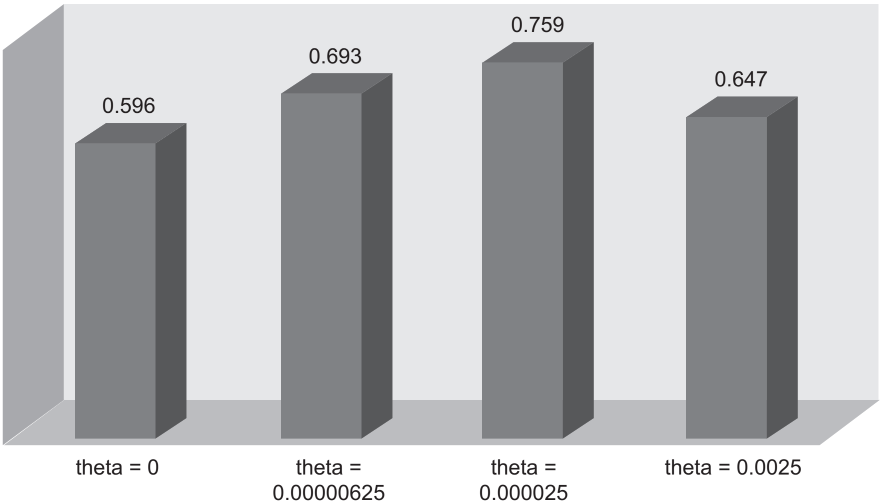

Figure 24.6 Information ratio when residual alpha is return and risk.

chart we plot the information ratio of the optimized portfolio, the information ratio is maximized when $\theta = 0.000025$, approximately equal to $\lambda\sigma^2_{\alpha_{R_\perp}}$, where $\sigma^2_{\alpha_{R_\perp}}$ is the variance of the return to the residual alpha. The portfolio will achieve the highest risk—reward tradeoff when θ associates the proper risk penalty with a bet on residual alpha. If θ is too big or too small, the volatility of the residual alpha will be over- or underestimated, resulting in a suboptimal portfolio.

24.7 Conclusion

Portfolio managers use different factor models for alpha and risk, and misalignment of those factors is a common phenomenon. Alpha can be decomposed into a part spanned by the risk model and a part orthogonal to it (the so-called orthogonal or residual alpha). Mitigation of alpha and risk factor misalignment may be approached at various levels. Alteration of the risk model, adjusting the optimization process and adding additional risk factors were some of the opportunities discussed. The tendency of optimizers to favor residual alpha is addressed by adding a penalty term to the utility function. We also showed that should the risk model be missing the orthogonal alpha as a risk factor, penalizing residual alpha is a superior mitigation for risk and alpha factor misalignment than the alpha factor approach.

References

Bender, J., 2008. Measuring the efficiency of portfolio construction. MSCI Barra Research Insights MSCI, New York.

Bender, J., Lee, J., Stefek, D., 2009. Refining portfolio construction when alphas and risk factors are misaligned. MSCI Barra Research Insights MSCI, New York.

Grinold, R.C., Kahn, R.N., 2008. Active *Portfolio Management*. McGraw-Hill Professional, New York.

Lee, J., Stefek, D., 2008. Do risk factors eat alphas. The Journal of Portfolio Management 34 (4), 12–25.

Saxena, A., Stubbs, R., 2010. Alpha alignment factor: a solution to the underestimation of risk for optimized active portfolios. Axioma Research Paper 15 Axioma, New York.

Saxena, A., Stubbs, R., 2011. Pushing the frontiers (literally) with the alpha alignment factor. In: Axioma Research Paper 22, Axioma, New York.

Stefek, D., 2007. Getting the most out of portfolio optimization—guarding against estimation error. MSCI Research Conference, New York.