Estimation of zero-coupon curves in DataMetrics

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DataMetrics is modifying its technique for estimating zero-coupon interbank and government benchmark curves. The new algorithm is employed together with additional synchronized input data to deliver better-quality curves. The modified technique assumes the instantaneous forward rate is a constant between the maturity dates of observable interest rates. Together, the flat forward technique and new input data increase pricing and risk measurement accuracy, particularly at the short end. The flat forward technique is shown to be preferable to plausible alternative approaches.

DataMetrics is modifying the bootstrapping technique it uses to estimate zero-coupon curves. The modified bootstrapping technique assumes **the instantaneous forward interest rate is a constant between observed bond or other security maturities**. We therefore refer to it as the "flat forward" technique. DataMetrics applies it to a range of interbank and government benchmark zero-coupon curves in both key currencies and emerging markets.

The new algorithm is being employed together with additional input data to deliver better-quality curves. Together, the flat forward technique and new input data provide several advantages:

- The underlying data for the interbank and government curves, particularly the government bonds, can be sparse. For example, there are only 6 U.S. Treasury benchmarks, of which 3 are T-bills and 3 are coupon bonds. The flat forward approach avoids potentially adding spurious information to that contained in the modest available underlying data set, while still permitting accurate pricing.

- It permits DataMetrics to estimate time series of spot interest rates with any time to maturity ranging from overnight to 30 years, applying any desired compounding interval or day count convention uniformly across the curve. In particular, it becomes it easy to add new RiskMetrics vertexes if needed and supported by the data. It also permits the generation of time series of forward rates with any compounding interval, time to maturity, or time to settlement.

- It increases the pricing accuracy of DataMetrics fixed-income curves for fixed-income instruments, particularly at the short end.

- The new approach improves the accuracy of volatility and correlation estimates and thus of fixed-income risk measures generally. It particularly enhances risk reporting for short-term fixed income positions and for money-market futures spread trades.

- The modified bootstrapping technique permits the original security prices used in the estimation procedure to be recovered from spot rates. For example, if a money market futures expiring in exactly 15 months is used in the procedure, the implied rate can be recovered from the estimated 15- and 18-month spot rates.

- The data underlying the curves can generally be captured simultaneously, so the new curves will be synchronous.

We can think of an interest-rate curve as an unobservable price schedule for fixed income securities with the same liquidity and credit-risk characteristics, but diverse cash flow structures and day count and compounding conventions. A government benchmark curve is the price schedule for central government obligations; an interbank curve is the price schedule for debt between highly-rate banks, or between banks and highly-rated corporate counterparties. Such price schedules should be expected to accurately reflect interest rates on any fixed-income security in that category, regardless of cash flow structure and day count and compounding conventions.

The flat forward approach fully exploits all the information about an interest rate curve contained in the observable market data, while at the same time adding no additional assumptions about the level and shape of the curve. The precise settlement practices, maturity dates, compounding frequency, pay frequency and day count conventions underlying the input data are taken into account in the calculations.[1]

1 Constructing the curve: source data

There are two aspects to the curve construction technique: how the raw data are manipulated, and how the raw data are selected from the many fixed-income securities on the curve. We will take them in turn in this and the next section.

Interbank or swap curves represent interest rates on unsecured, non-negotiable loans between high-quality banks and corporations. They can be based on interbank deposit rates or fixings, money market futures, and plain vanilla swap rates. The shortest maturity points on the interbank curve are generally derived from indicative deposit rates or deposit rate fixings. Where available, the first point on the curve is an overnight rate, generally an overnight Libor or repo rate. For the U.S. dollar (effective Fed funds) and the Euro (Eonia), a weighted average index of overnight transactions are available. For emerging market currencies and some industrial country currencies, overnight index swap rates and money market rates implied by foreign exchange forwards may be used. Where available, money market futures are used for the 3-month to 2- or 3-year sector, and swap rates for the longer-term sector.

Government benchmark curves represent default risk-free interest rates. Central or federal government bills and bonds are used where possible. In some cases, liquid markets in short-term central government debt do not exist. In that case, interbank obligations may be used to construct the short end of the curve.

[1] Further detail on the flat forward technique is contained in Malz (2002). Earlier work on DataMetrics yield curves is presented in Zangari (1997). Fabozzi (1999) has a textbook description of the standard bootstrapping approach to yield curve estimation. A recent example of the use of the flat forward technique, by the European Central Bank, is described in Brousseau (2002).

Table 1
Interbank curve data sources by maturity range

U.S. dollar		
Overnight	Effective Federal funds rate or USD deposits	Actual/360
1 week to 3 months	USD deposits	Actual/360
3 months to 2 years	Settlement prices of Chicago Mercantile Exchange (CME) 3-month Eurodollar futures	Actual/360
2 years to 30 years	Plain vanilla interest rate swaps vs. 6-month Libor flat	Semiannual bond basis
Euro		
Overnight	EONIA or EUR deposits	Actual/360
1 week to 3 months	EUR deposits	Actual/360
3 months to 2 years	Settlement prices of London International Financial Futures and Options Exchange (LIFFE) 3-month Euribor futures	Actual/360
2 years to 30 years	Plain vanilla interest rate swaps vs. 6-month Libor flat	Annual bond basis

The source data generally has different day count conventions. All rates are converted if necessary to a common day count convention of Actual/365 as part of the curve estimation procedure. For money market rates and rates implied by 3-month money market futures, this generally involves multiplying the rates by $\frac{360}{365}$. Futures on 3-month money market rates are treated as having a time to maturity of exactly $\frac{91}{365}$ years. Regardless of the day count basis or other bond-mathematical conventions underlying the input data, the output data can be converted to any desired basis. DataMetrics and RiskManager employ Actual/365 as a common default day count basis for all curves.

In many cases, settlement of fixed-income transactions occurs one or several days after the counterparties conclude a deal. For example, the value date for most OTC money market transactions is 2 days after the transaction date. We take the time to settlement into account in identifying the maturity date correctly for money market transactions less than 1 year. However, we do not take the time to settlement itself into account in setting discount factors and zero-coupon rates for two reasons. First, in most cases, the time to settlement is not uniform across the data sources underlying the curves. Second, and more importantly, the time to settlement does not affect the validity of the interest rate as a representation of today's price of money for the term to maturity, as interest is not charged until the value date.

Details are presented in Table 1 for the USD and EUR interbank curves.

2 Constructing the curve: mechanics

Each curve has a short-maturity segment based on add-on or discount rates or forwards or futures on add-on or discount rates. This sector of the curve, called the "stub," does not require bootstrapping, as the rates on the underlying instruments can be readily converted to zero-coupon or spot rates. Longer term rates are based on coupon bonds or swaps. The longer-term portion of each curve is bootstrapped using the flat forward assumption. The precise settlement practices, maturity dates, compounding frequency, pay frequency and day count conventions underlying the input data are taken into account in the calculations.

2.1 Notation

We let t denote the current date, T and its subscripted variants a maturity, settlement or payment date, and τ and its subscripted variants, measured in years, a time to maturity, settlement or payment.

The immediate output of the procedure is a set of instantaneous forward rates with specific settlement dates, which we call **forward vertexes**, as shown in Figure 1. Let T_j, $j = 0, 1, ..., N$ be the forward vertexes for which instantaneous forward rates are estimated, and let $\{f_{\tau_j}\}_{j=1}^{N}$ be the set of estimated instantaneous forward rates from which the curve is constructed. The indexing is carried out so that each instantaneous forward rate f_{τ_j} applies on the interval $[\tau_{j-1}, \tau_j)$. The flat forward assumption implies that for any $0 \leq \tau \leq \tau_N$, $f_\tau = f_{\min[\{\tau_j | \tau_j > \tau\}]}$. As a convention, we set $\tau_0 \equiv 0$; for $\tau \geq \tau_N$, we set $f_\tau = f_{\tau_N}$.

2.2 Relationships among spot and forward rates

A τ-year continuously compounded spot rate, that is, the constant annual rate at which a pure discount bond's value must grow to reach one currency unit at time T, is computed by integrating the instantaneous forward curve over the time to maturity:

$$r_\tau = \frac{1}{\tau} \int_0^\tau f_s ds. \tag{1}$$

If the instantaneous forward rate is higher (lower) than the spot rate for maturity τ, the spot rate rises (falls) at the exponential rate $\frac{1}{\tau}(f_\tau - r_\tau)$, as shown in Figure 1.

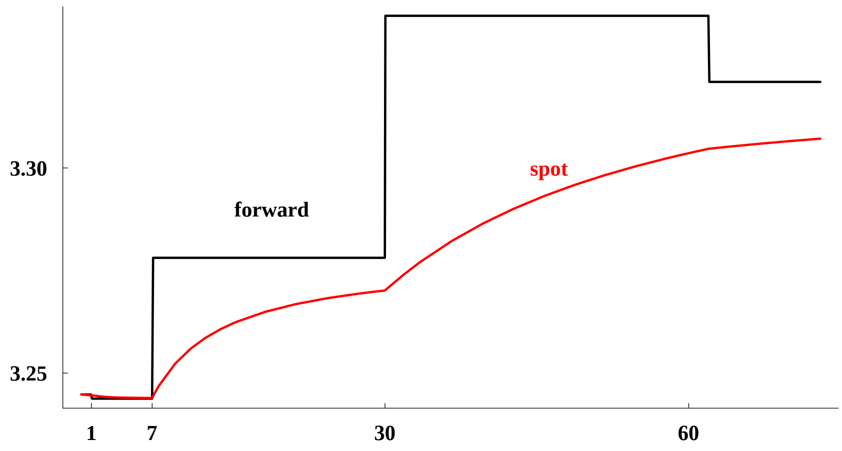
Discount factors, that is, the time-t prices of pure discount bonds maturing at time T are constructed from the forward rates or spot rates. The discount factor p_τ is related to spot rates by

$$p_\tau = e^{-r_\tau \tau} = \exp\left(-\int_0^\tau f_s ds\right). \tag{2}$$

2.3 Constructing the stub portion

Deposit rates are expressed as annual add-on rates, that is, interest is computed by multiplying the quoted rate by the fraction of a year for which interest will be applied. Let $r_{t,T}^{\text{dep}}$ represent a $\tau \equiv T - t$ deposit rate. Its discrete compounding interval is equal to $t - T = \tau$. The time to maturity of a deposit or fixing is computed from the actual maturity date, which in turn

Figure 1
Forward and spot rates



Instantaneous forward rates and continuously compounded spot rates, EUR swap curve, Nov. 16, 2001.

is computed using the ISDA Modified Following Business Day convention and its refinements for maturities below 1 month.

The continuously compounded spot rate $r_{t,T}$ is

$$r_{t,T} = \frac{\ln(1 + r^{\text{dep}}_{t,T}\tau)}{\tau}. \tag{3}$$

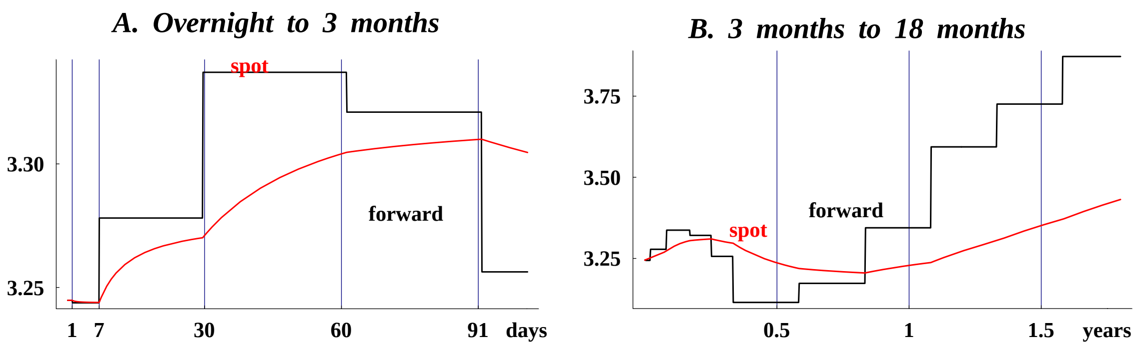
A continuously compounded forward rate f_{t,T_1,T_2} with time to settlement τ_1 and time to maturity τ_2 can be computed from the continuously compounded spot rates derived from two deposit rates with maturities T_1 and T_2. Letting $T_1 - t = \tau_1$ and $T_2 - T_1 = \tau_2$,

$$f_{t,T_1,T_2} = \frac{r_{t,T_2}(\tau_1 + \tau_2) + r_{t,T_1}\tau_1}{\tau_2}. \tag{4}$$

We need next to convert the forward with a discrete time to maturity (τ_2) to an instantaneous forward curve. A forward rate with a discrete time to maturity is the average of the instantaneous forward rates over the time to maturity:

$$f_{t,T_1,T_2} = \frac{1}{\tau_2}\int_{T_1}^{T_2} f_{t,s}\,ds. \tag{5}$$

Figure 2
Forward and spot rates: overnight to 18 months



Instantaneous forward rates and continuously compounded spot rates, EUR swap curve, Nov. 16, 2001. Vertical grid lines represent DataMetrics cash flow vertexes.

Under the flat forward assumption, the instantaneous forward rate $f_{t,T}$ is constant over the interval $T_1 \leq T \leq T_2$, conveniently implying

$$f_{t,T} = f_{t,T_1,T_2}, \qquad T_1 \leq T \leq T_2. \tag{6}$$

The computations for the Libor-based short end are illustrated for the EUR swap curve in Panel A of Figure 2.

To apply this to government bills, let $r^{\text{bill}}_{t,T}$ be the bond-equivalent yields to maturity on Treasury bills. For U.S. T-bills, the convention is to quote the rates as if they were deposit rates (no compounding) for remaining times to maturity under 182 days and as spot rates with semi-annual compounding for maturities 6 months and greater.

For interbank curves, the next portion of the curve is computed for major currencies from prices of money market futures. Let $t - T_1 = \tau_1$ be the time to maturity of the futures contract and $T_2 - T_1 = \tau_2$ the time to maturity of the underlying deposit fixing rate. We treat the money market rate f_{t,T_1,T_2,τ_2} implied by the futures price as a money market forward with time to settlement τ_1 and time to maturity and compounding interval equal to τ_2. We ignore the convexity adjustment arising from the mark-to-market and margining features of futures contracts, as it is likely to be very small for the relatively short futures maturities used here.

Most money market futures are claims on 3-month money market rates, so τ_2 is the time to maturity of the underlying 3-month deposit and is expressed on an Actual/360 or Actual/365 basis, depending on currency. We convert to Actual/365 when that is not the day count basis for the currency in question, but we do not compute the exact time to maturity, instead assuming a uniform time to maturity $\tau_2 = \frac{91}{365}$ years. This assumption induces at worst a trivial distortion when the actual number of days to maturity is, say, 89 or 93.

The continuously compounded forward rate $f_{\tau_1,\tau_1+\tau_2}$ with time to settlement τ_1 and time to maturity τ_2 is

$$f_{\tau_1,\tau_1+\tau_2} = \frac{\ln(1 + f_{\tau_1,\tau_1+\tau_2,\tau_2}\tau_2)}{\tau_2}. \tag{7}$$

Just as in the case of two money market rates with different maturities, Equation (6) implies that this is the instantaneous forward rate over the interval $\tau_1 \leq \tau < \tau_2$.

The computations for the futures-based portion of the curve are illustrated for the EUR swap curve in Panel B of Figure 2.

2.4 Constructing the coupon portion

The portion of the curve based on swap rates or coupon bonds is derived by bootstrapping. The procedure is based on the bond equation

$$p_{t,T,h}(c) = c \cdot h \sum_{k=1}^{K} e^{-r_{t,\tau_k}\tau_k} + e^{-r_{t,T}\tau}, \tag{8}$$

where $p_{t,T,h}(c)$ is the dirty price of a bond maturing at time T with an annual coupon rate c, expressed as a decimal and paid $\frac{1}{h}$ times annually, and $\tau_k = T_k - t, k = 1, ...K$, with $T_K = T$, are the times to the coupon or fixed-rate payments. The bond price is expressed as a percent of par, i.e. par equals unity. We can express the bond price in terms of discount factors or instantaneous forward rates rather than spot rates by substituting Equation (1) or Equation (2) into Equation (8).

A similar but somewhat simpler equation relates quoted swap rates to the par value of a coupon bond with a coupon equal to the swap rate:

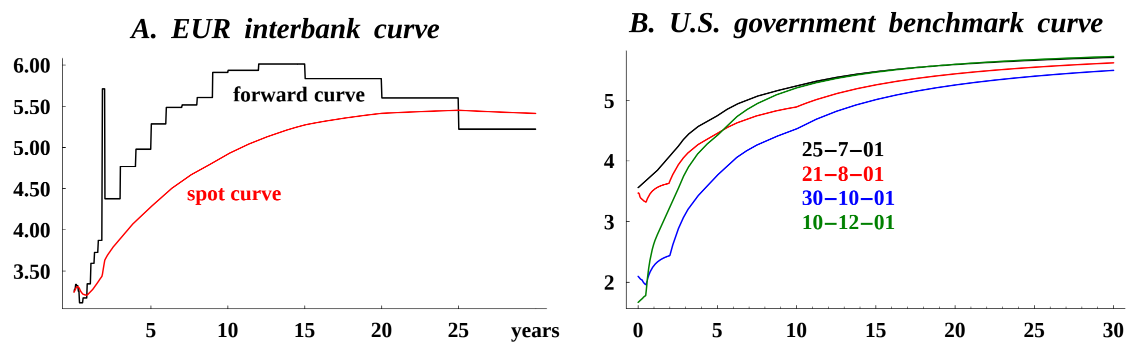
$$1 = c \cdot h \sum_{k=1}^{\frac{\tau}{h}} e^{-r_{t,k\cdot h}k\cdot h} + e^{-r_{t,T}\tau}, \tag{9}$$

In bootstrapping, the $p_{t,T,h}(c)$ are observed, and we use Equation (8) to infer the unobserved $r_{t,T}$. The particular trick in bootstrapping is to sequence the computations so that only a subset of $r_{t,T}$ (or corresponding subset of $p_{t,T}$ or $f_{t,T}$) at a time appears as an unknown in Equation (8).

In our version of bootstrapping, we employ the **forward discount factor** p_{t,T_1,T_2}, that is, the time-t price of a forward claim on a discount bond settling at time T_1 and maturing (at a value of one currency unit) T_2. It is related to spot discount factors by

$$p_{t,T_1,T_2} = \frac{p_{t,T_2}}{p_{t,T_1}}. \tag{10}$$

Figure 3
Estimated spot and forward rates



RiskMetrics volatilities with a 90-day window in percent at an annual rate. Volatility of discount factor ("price volatility" or "swap zero volatility").

We can also express the forward discount factor in terms of instantaneous forward rates:

$$p_{t,T_1,T_2} = \frac{\exp\left(-\int_0^{T_2} f_{t,s} ds\right)}{\exp\left(-\int_0^{T_1} f_{t,s} ds\right)} = \exp\left(-\int_{T_1}^{T_2} f_{t,s} ds\right). \tag{11}$$

If the instantaneous forward rate is constant between settlement dates T_1 and T_2, we have

$$p_{t,T_1,T_2} = e^{-f_{t,T_2}\tau_2}. \tag{12}$$

The bootstrapping step is to solve Equation (12) for f_{t,T_2}. Numerically, this is similar to a yield to maturity calculation and is well-behaved.

Estimates for the entire curve and for the U.S. government benchmark curve are displayed in Figure 3.

2.5 Creating constant-maturity time series on a common compounding and day count basis

Before presenting the zero-coupon rates, a set of bond-mathematical conventions and a set of **cash flow nodes** or **vertexes** must be determined. The risk engine must be aware of these settings when the data is presented. Cash flow vertexes are the times to maturity of constant-maturity time series of spot rates or discount factors. For each curve, the constant-maturity spot rate time series are the risk factors to which fixed-income exposures of the corresponding type are mapped.[2] Spot and forward rates with specific discrete compounding intervals can now be constructed from discount factors or continuously compounded spot rates.

Cash flow and forward vertexes are not generally identical. The interest amount over the interval $[t, T]$ is $r_{t,T}\tau$. We can infer only the incremental interest amount $f_{t,T_{j+1}}(T_{j+1}-T_j)$ accruing—or equivalently, the average rate at which interest accrues—between successive forward vertexes T_{j+1} and T_j from the observed data. The hypothesis that the forward rate is constant between vertexes thus corresponds precisely to the actual limitations of the data. The forward rate for a forward vertex T_{j+1} is computed so as to exactly match each successive observed security price, given the set of forward rates $f_{t,T_k}, k = 0, \ldots, j$. Original security prices cannot generally be reconstructed from spot rates for cash flow vertexes.

The spot rates corresponding to each cash flow vertex are almost always found by interpolation, since the forward vertexes are generally close to but not precisely equal to the desired cash flow vertexes. The observed data do not impose a choice of vertexes, since spot rates for any time to maturity can be computed from the forward vertexes. However, by choosing vertexes that are close to the observed data on which a particular curve is based, we can avoid creating risk factors in which interpolation plays more than a minor role, while still being able to map most exposures to relatively nearby vertexes.

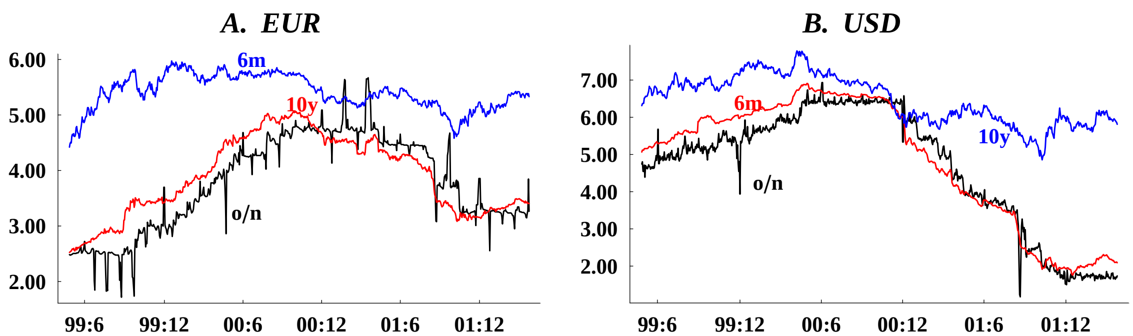
Figure 4 displays time series of spot rates for the period October 1998 to October 2000. The 1-month rate rises at the end of 1998 and 1999: year-end spikes in short rates are routine. The 6- and 18-month rate were less susceptible to Y2K effects. The modified bootstrapping technique eliminated the need to edit data in the runup to Y2K.

For the Euro during 1999, there are frequent spikes, mostly downward, in the overnight rate. These spikes generally occur on the 23rd day of each month, the terminal date for the measurement of reserve balances and reservable liabilities in the European Central Bank (ECB), and have become less pronounced as the European banking system gained experience with ECB operating procedures. There is a similar residue of central bank operating procedures on very short-term USD rates, but less pronounced: average reserves and reserve requirements for the 2-week reserve maintenance period are calculated on "settlement Wednesdays." For both currencies, there are also day-of-week effects. In particular, overnight rates tend to fall on Fridays, as banks shed excess balances that will earn low rates over a 3- or 4-day period.

Figure 6 displays correlations of discount factors along the curve over time for the period October 1998 to October 2000. The correlation between the 1-year and 2-year interbank discount factor is difficult to estimate for most currencies using most estimation techniques, since this interval encompasses the "graft point" at which the input data switch from deposits, forward rate agreements or money market futures to plain vanilla swaps. Using the flat forward approach, estimates of this correlation are high and remain fairly high (above 0.65) even at year-end.

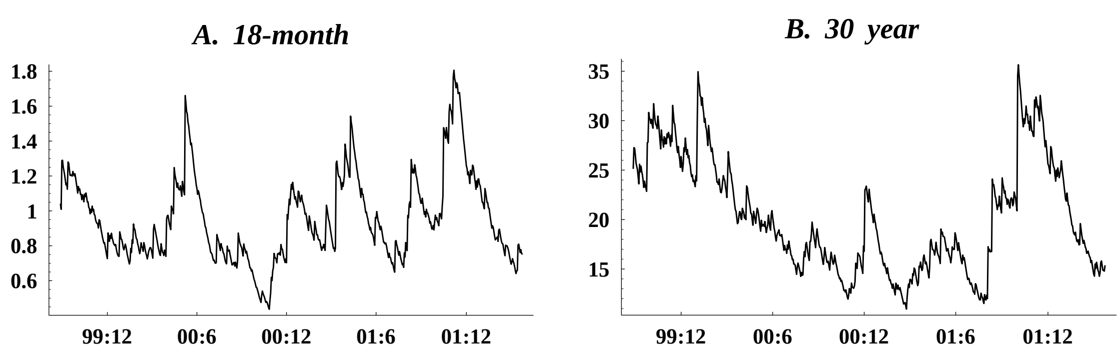
[2] Mina and Xiao (2001), in their discussion of cash-flow mapping (pp. 41ff.), refer to cash flow vertexes as RiskMetrics vetexes.

Figure 4
Estimated interbank spot rates



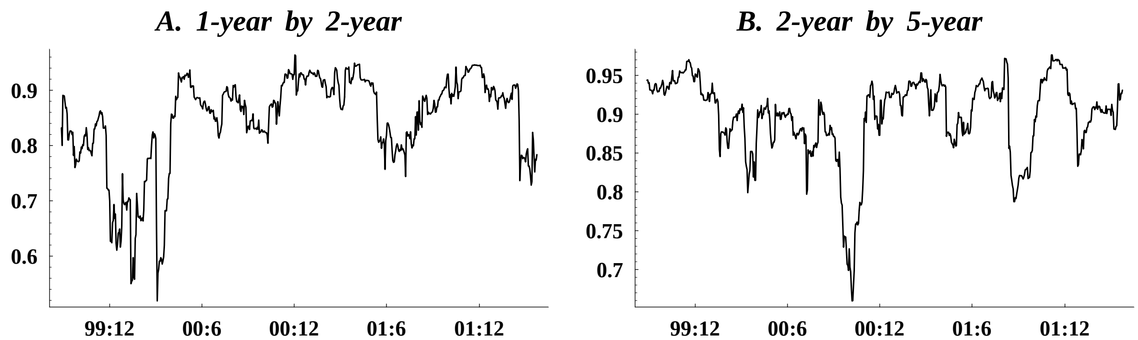
Estimated spot rates, May 26, 1999 to April 24, 2002. O/n: overnight. Source: DataMetrics.

Figure 5
Estimated interbank volatilities



RiskMetrics volatilities with a 90-day window in percent at an annual rate. Volatility of discount factor (“price volatility” or “swap zero volatility”).

Figure 6
Estimated interbank correlations



RiskMetrics correlations with a 90-day window at an annual rate. Correlations of discount factors.

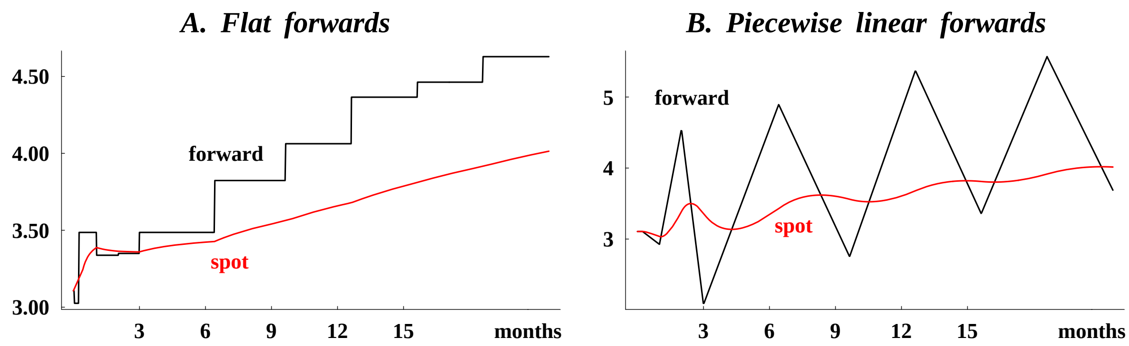
3 Alternative approaches are less satisfactory

An alternative to the flat forward assumption is to assume that the instantaneous forward rate is continuous and linear between forward vertexes, that is, the instantaneous forward curve. The piecewise linear approach is appealing for several reasons:

- It permits the forward curve to be a continuous function rather than a discontinuous step function.
- Under the flat forward assumption, the spot curve is not continuously differentiable. This manifests itself in kinks in the spot curve at the forward vertexes. If the forward curve is above the spot rate, so the spot rate is rising, the spot curve is concave to the origin. If the forward curve is below the spot rate, so the spot rate is falling, the spot curve is convex to the origin.
- At the same time, it retains the advantage of the flat forward approach of not assuming anything about the curve beyond what is contained in observable market prices.

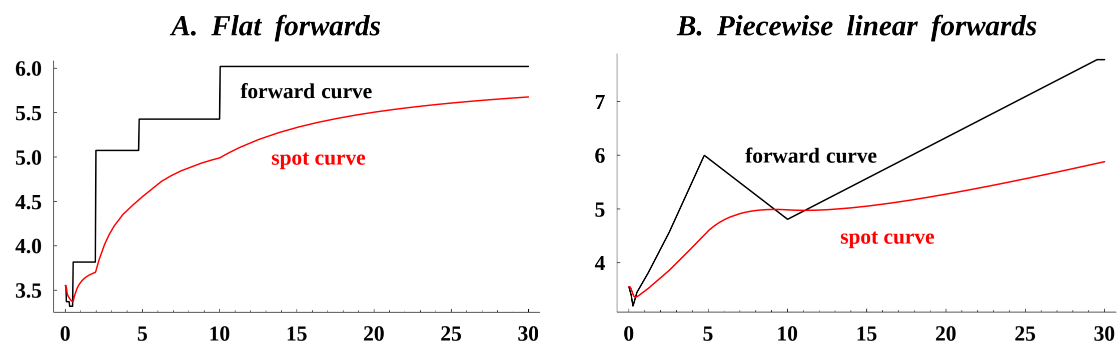
Unfortunately, the piecewise linear approach can induce wide swings in forward rates when the spot curve is not monotonic or when it has strong variations in slope between forward vertexes. This property is illustrated in Figure 7, which compares the two approaches for the stub portion of the EUR interbank curve. The spot rate rises by about 50 basis points over the 6 month to 2 year maturity interval. Under the flat forward assumption, the estimated forward rates rise somewhat faster, about 100 basis points, and monotonically. Under the piecewise linear forwards assumption, the forward rates swing up and down with an amplitude of about 200 basis points, but rising on net over the 6 month to 2 year maturity interval. The spot rates for the forward vertexes are identical under both approaches. The resulting spot curve under the piecewise linear forwards assumption is smoother, but has wiggles that are difficult to interpret. The swings in the forward curve are difficult to accept.

Figure 7
Model comparison: EUR interbank rates



Continuously compounded spot rates and instantaneous forward rates, Nov. 30, 1999.

Figure 8
Model comparison: US government benchmark rates



Continuously compounded spot rates and instantaneous forward rates, Apr. 2, 2001.

Figure 8 compares the two approaches for the U.S. government benchmark curve. The raw data is more sparse, so the concavity of the spot curve between forward vertexes is more pronounced than for the interbank curves. The spot rate is rising over most of the term structure, but the upward slope diminishes at the long end. The flat forward approach captures this behavior with much smaller variations in the forward rate. Under the piecewise linear forwards assumption, the forward rate is forced to drop and then rise precipitously in order to price the observable securities precisely.

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